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Bianchi V Models in $N=2$, $D=5$ Supergravity

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We investigate Bianchi V cosmological models containing two interacting scalar fields. These models are derived from a dimensional reduction of the $N=2$, $D=5$ supergravity theory. Exact solutions are found.

1. INTRODUCTION

In recent years there has been intensive study of physical theories in more than four dimensions, among them strings, superstrings, and supergravity. Balbinot *et al.* (1990a,b) considered cosmological solutions for the $N=2$ and $D=5$ supergravity theory (D'Auria *et al.*, 1982; Cremmer, 1980), which contains the metric g_{MN}, a spin $-3/2$ field ψ^a_M ($a = 1, 2$ is an internal index), and a 1-form B_M ($M, N = 1, \ldots, 5$ and $\mu\nu = 1, \ldots, 4$). Looking for a "ground-state" configuration, they set the fermion field and the electromagnetic field in the metric g_{MN} equal to zero. They assumed a local structure $V_4 \times S^1$, where V_4 is a four-dimensional spacetime, and they found exact solutions when V_4 is a Robertson–Walker spacetime; their solutions are nonsingular. In a previous paper Pimentel (1992) considered the case when v_4 is a Bianchi type I manifold, and obtained exact solutions in which the singularity is not avoided. In this paper we will consider the Bianchi type V spacetime. This is an anisotropic but homogeneous spacetime that has been used before as a cosmological model. This model represents a generalized Friedmann–Robertson–Walker cosmology with negative spatial curvature ($k = -1$). Although Bianchi type $\mathrm{VI_h}$ is also a generalization of the negative Robertson–Walker model, in the case of Bianchi type V Einstein's equations are more tractable.

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2. FIELD EQUATIONS

The theory studied by Balbinot *et al.* (1990a) has the five-dimensional line element

$$dS^2 = g_{\mu\nu}(x^\mu)dx^\mu dx^\nu - \phi^2(x^\mu)(dx^5)^2 = ds^2 - \phi^2(x^\mu)(dx^5)^2 \tag{1}$$

It is also assumed that the 1-form B_M is $B_M = (0, 0, 0, 0, \psi(x^\mu))$. With all these assumptions the theory is equivalent to one with a four-dimensional Lagrangian given by

$$\mathcal{L} = \frac{1}{4}\sqrt{g}\phi\left(R + 2\frac{D_\lambda\psi D^\lambda\psi}{\phi^2}\right) \tag{2}$$

The field equations obtained from the variation of the above Lagrangian are

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{3}{2\phi^2}\left(\psi_{,\mu}\psi_{,\nu} - \frac{1}{2}g_{\mu\nu}\psi_{,\lambda}\psi^{,\lambda}\right) + \frac{1}{\phi}(\phi_{;\mu\nu} - g_{\mu\nu}\Box\phi) \tag{3}$$

$$\Box\phi + \frac{\psi_\lambda\psi^\lambda}{\phi} = 0 \tag{4}$$

$$\Box\psi - \frac{\phi_\lambda\psi^\lambda}{\phi} = 0 \tag{5}$$

In this section we set the field equations for a Bianchi type V metric. We use the following metric:

$$ds^2 = -dt^2 + e^{2A}dx^2 + e^{2B+2qx}dy^2 + e^{2C+2qx}dz^2 \tag{6}$$

where A, B, and C are functions of the cosmological time t, and q is a parameter. This metric reduces to the Friedmann–Robertson–Walker metric with $k = -1$ when $A = B = C$ and $q = 1$. The field equations for this particular spacetime are given by

$$\dot{A}\dot{B} + \dot{A}\dot{C} + \dot{B}\dot{C} - 3q^2e^{-2A} = -(A + B + C)^{\cdot}\left(\frac{\dot{\phi}}{\phi}\right) + \frac{3}{4}\left(\frac{\dot{\psi}}{\phi}\right)^2 \tag{7}$$

$$\ddot{B} + \dot{B}^2 + \ddot{C} + \dot{C}^2 + \dot{B}\dot{C} - q^2e^{-2A} = -(B + C)^{\cdot}\left(\frac{\dot{\phi}}{\phi}\right) - \left(\frac{\ddot{\phi}}{\phi}\right) - \frac{3}{4}\left(\frac{\dot{\psi}}{\phi}\right)^2 \tag{8}$$

$$\ddot{A} + \dot{A}^2 + \ddot{C} + \dot{C}^2 + \dot{A}\dot{C} - q^2e^{-2A} = -(A + C)^{\cdot}\left(\frac{\dot{\phi}}{\phi}\right) - \left(\frac{\ddot{\phi}}{\phi}\right) - \frac{3}{4}\left(\frac{\dot{\psi}}{\phi}\right)^2 \tag{9}$$

$$\ddot{A} + \dot{A}^2 + \ddot{B} + \dot{B}^2 + \dot{A}\dot{B} - q^2e^{-2A} = -(A + B)^{\cdot}\left(\frac{\dot{\phi}}{\phi}\right) - \left(\frac{\ddot{\phi}}{\phi}\right) - \frac{3}{4}\left(\frac{\dot{\psi}}{\phi}\right)^2 \tag{10}$$

$$2\dot{A} = \dot{B} + \dot{C} \tag{11}$$

and

$$\ddot{\phi} + (A + B + C)\dot{}\dot{\phi} + \frac{\dot{\psi}^2}{\phi} = 0 \tag{12}$$

$$\ddot{\psi} + (A + B + C)\dot{}\dot{\psi} - \frac{\dot{\psi}\dot{\phi}}{\phi} = 0 \tag{13}$$

Here the dot means derivative with respect to t. In the following section we look for exact solutions to the system of differential equations formed by equations (7)–(13).

3. EXACT SOLUTIONS

In order to put the above system in a more symmetric form we define a new time variable by $d\eta = \exp(-A)\, dt$, and now equations (7)–(13) are given by

$$A'B' + A'C' + B'C' - 3q^2 = -(A + B + C)' \frac{\phi'}{\phi} + \frac{3}{4}\left(\frac{\psi'}{\phi}\right)^2 \tag{14}$$

$$B'' - A'B' + B'^2 + C'' - A'C' + C'^2 + B'C' - q^2$$
$$= (A - B - C)' \frac{\phi'}{\phi} - \frac{\phi''}{\phi} - \frac{3}{4}\left(\frac{\psi'}{\phi}\right)^2 \tag{15}$$

$$A'' + C'' + C'^2 - q^2 = -C' \frac{\phi'}{\phi} - \frac{\phi''}{\phi} - \frac{3}{4}\left(\frac{\psi'}{\phi}\right)^2 \tag{16}$$

$$A'' + B'' + B'^2 - q^2 = -B' \frac{\phi'}{\phi} - \frac{\phi''}{\phi} - \frac{3}{4}\left(\frac{\psi'}{\phi}\right)^2 \tag{17}$$

$$2A' = B' + C' \tag{18}$$

and

$$\phi'' + (B + C)'\phi' + \frac{\psi'^2}{\phi} = 0 \tag{19}$$

$$\psi'' + (B + C)'\psi' - \frac{\psi'\phi'}{\phi} = 0 \tag{20}$$

Here the prime means the derivative with respect to η. Multiplying equation (19) by ϕ' and equation (20) by ψ' and adding, we obtain

$$\frac{1}{2}(\phi'^2 + \psi'^2)' + 2A'(\phi'^2 + \psi'^2) = 0 \tag{21a}$$

where we have used equation (18). A first integration of this equation gives

$$\phi'^2 + \psi'^2 = \alpha_1 e^{-4A}, \qquad \alpha_1 > 0 \tag{21b}$$

Equation (20) implies that

$$\psi' = \alpha_2 e^{-2A}\phi \tag{22}$$

Adding equations (14) and (15), we have

$$(\phi'' + 2A''\phi + 4A'\phi' + 4A'^2\phi)e^{2A} = 4q^2\phi e^{2A} \tag{23a}$$

but this is

$$(\phi e^{2A})'' = 4q^2\phi e^{2A} \tag{23b}$$

from which we have

$$\phi e^{2A} = \alpha_3 e^{2q\eta} + \alpha_4 e^{-2q\eta} \tag{23c}$$

with α_3 and α_4 integration constants. Eliminating A from equations (21b) and (22), we obtain

$$\psi' = \pm\frac{\alpha_2\phi\phi'}{(\alpha_1 - \alpha_2^2\phi^2)^{1/2}} \tag{24a}$$

from which we have

$$\psi = \psi_0 \mp \frac{1}{\alpha_2}(\alpha_1 - \alpha_2^2\phi^2)^{1/2} \tag{24b}$$

Using equations (22), (24a), and (23c) to eliminate e^{2A}, we get the following equation for ϕ:

$$\frac{\phi'}{\phi(\alpha_1 - \alpha_2^2\phi^2)^{1/2}} = \frac{\pm 1}{\alpha_3 e^{2q\eta} + \alpha_4 e^{-2q\eta}} \tag{25}$$

The solution to this equation is given by

$$\phi = \frac{2\sqrt{\alpha_1}}{\alpha_2}\,\frac{\alpha_5 \exp(-I\sqrt{\alpha_1})}{1 + \alpha_5^2 \exp(-2I\sqrt{\alpha_1})} \tag{26a}$$

Here α_5 is an integration constant and

$$I(\eta) = \int^{\eta} \frac{1}{\alpha_3 e^{2q\eta} + \alpha_4 e^{-2q\eta}} \tag{26b}$$

The explicit value of this integral will be postponed, since it depends on the sign of the product of α_3 and α_4 or if one of them vanishes. Subtracting equation (16) from equation (17), we have

$$C'' - B'' + C'^2 - B'^2 = (-C' + B')\left(\frac{\phi'}{\phi}\right) \tag{27}$$

but this equation has the first integral

$$(C - B)'\phi = \alpha_6 e^{-2A} \tag{28}$$

Using equation (23c) to eliminate A and integrating, we obtain

$$C - B = \alpha_6 I(\eta) \tag{29}$$

From equation (24b) it is clear that

$$\psi = \psi_0 - \frac{\sqrt{\alpha_1}}{\alpha_2}\frac{1 - \alpha_3(\tanh q\eta)^{\alpha_1^{1/2}/qk}}{1 + \alpha_3(\tanh q\eta)^{\alpha_1^{1/2}/qk}} \tag{30}$$

We obtain the solution for A from equation (23c)

$$A = \frac{1}{2}\ln\left[\frac{\alpha_3 e^{2q\eta} + \alpha_4 e^{-2q\eta}}{\phi}\right] \tag{31}$$

From the equation $C' + B' = 2A'$ we obtain

$$C + B = 2A \tag{32}$$

where, without loss of generality, we have set an additive integration constant equal to zero. From Eqs. (29)–(32) we obtain the solutions for B and C,

$$B = A - \alpha_6 I(\eta) = \frac{1}{2}\ln\left[\frac{\alpha_3 e^{2q\eta} + \alpha_4 e^{-2q\eta}}{\phi}\right] - \alpha_6 I(\eta) \tag{33}$$

$$C = A + \alpha_6 I(\eta) = \frac{1}{2}\ln\left[\frac{\alpha_3 e^{2q\eta} + \alpha_4 e^{-2q\eta}}{\phi}\right] + \alpha_6 I(\eta) \tag{34}$$

If we now substitute the above expressions for A, B, C, ϕ, and ψ in the equations (14)–(17), we obtain the following relation among the integration constants:

$$\alpha_1 = -(4\alpha_6^2 + 48\alpha_3\alpha_4)/3 \tag{35}$$

Since $\alpha_1 > 0$ it is clear that $\alpha_3\alpha_4 < 0$ and now we can evaluate the integral

$I(\eta)$ and all the other functions that depend on it. The final form of the solutions is

$$I(\eta) = \frac{1}{4q(-\alpha_3\alpha_4)^{1/2}} \ln\left(\frac{e^{2q\eta} - (-\alpha_4/\alpha_3)^{1/2}}{e^{2q\eta} + (-\alpha_4/\alpha_3)^{1/2}}\right) \tag{36}$$

$$\begin{aligned} e^{2A} &= \frac{\alpha_2}{2\alpha_5\sqrt{\alpha_1}} (\alpha_3 e^{2q\eta} + \alpha_4 e^{-2q\eta}) \left(\frac{e^{2q\eta} - (-\alpha_4/\alpha_3)^{1/2}}{e^{2q\eta} + (-\alpha_4/\alpha_3)^{1/2}}\right)^{-\alpha_1/4q(-\alpha_1\alpha_3\alpha_4)^{1/2}} \\ &\quad \times \left[\alpha_5^2 + \left(\frac{e^{2q\eta} - (-\alpha_4/\alpha_3)^{1/2}}{e^{2q\eta} + (-\alpha_4/\alpha_3)^{1/2}}\right)^{\alpha_1/2q(-\alpha_1\alpha_3\alpha_4)^{1/2}}\right] \end{aligned} \tag{37}$$

$$e^{2B} = e^{2A}\left(\frac{e^{2q\eta} - (-\alpha_4/\alpha_3)^{1/2}}{e^{2q\eta} + (-\alpha_4/\alpha_3)^{1/2}}\right)^{-\alpha_6/2q(-\alpha_3\alpha_4)^{1/2}} \tag{38}$$

$$e^{2C} = e^{2A}\left(\frac{e^{2q\eta} - (-\alpha_4/\alpha_3)^{1/2}}{e^{2q\eta} + (-\alpha_4/\alpha_3)^{1/2}}\right)^{+\alpha_6/2q(-\alpha_3\alpha_4)^{1/2}} \tag{39}$$

With this we have been able to solve exactly the field equations for the case of Bianchi type V metric with two interacting scalar fields that come from a dimensional reduction of the $N=2$, $D=5$ supergravity theory.

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